Physics 139 - Relativity
Problem Set 10 Due Week April 3, 2003

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Classical Tests of General Relativity

Show your development of the solution by equations and diagrams before the evaluation to indicate your understanding of the problem.

1 Precession of the Perihelion of Mercury

The geodesic equation

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \tag{1}$$

for the Schwartzschild metric produces four second-order differential equations for the motion of a test particle in the field of a spherically symmetric mass.

$$\frac{d^2r}{ds^2} + \frac{1}{2}.... \tag{2}$$

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} - sin\theta cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0 \tag{3}$$

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} + 2cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} = 0 \tag{4}$$

$$c^2\frac{d^2t}{ds^2} + \frac{d\nu}{dr}\frac{dr}{ds}\frac{dt}{ds} = 0 \tag{5}$$

Chose a path (orbit) which moves along $\theta = \pi/2$ with $d\theta/ds = 0$. In that case $d^2\theta/ds^2 = 0$ and $\theta = \pi/2$ permanently. This is motion in an equatorial plane. Then with the 2nd equation solved we move to the third:

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} = 0 \tag{6}$$

which may be integrated once to yield

$$r^2\frac{d\phi}{ds} = constant = \ell \tag{7}$$

The first equation of motion becomes

$$\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2 - \frac{2GM}{c^2 r}\left[1 + r^2\left(\frac{d\phi}{ds}\right)^2\right] = k^2 - 1 \tag{8}$$

with r, ϕ being the cordinates of the Schwartzschild line element and ds is the proper time on a clock moving with the test body.

The similar Newtonian equations are:

$$r^2\frac{d\phi}{dt} = constant = \ell \tag{9}$$

and

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 - \frac{2GM}{c^2 r} = constant \tag{10}$$

where r, ϕ are ordinary polar coordinates and dt is the Newtonian (pre-relativistic) time.

Combine these two relativistic equations to show that the relativistic equation of motion is

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{c^2\ell^2} + 3GMu^2/c^2 \tag{11}$$

where $u = 1/r$ and this equation is the same as the Newtonian one except for the added $3GMu^2/c^2$

From the result of a previous problem set (relativistic hydrogen atom) take the solution and show the precession of Mercury in seconds of arc per century. Some useful numbers, if you do not have them from the previous problem set: Mercury Perihelion distance $= 45.9 \times 10^6$ km, the semimajor axis $a = 0.387094 A.U. = 1.45979 \times 10^8$ km, the eccentricity is 0.206 and the period is 0.241 years.

This is also provides a convenient way to calculate the gravitational deflection of light by a spherically symmetric mass. First note that the angular momentum forces $GM/c^2\ell^2 \to 0$. Then successive approximation. Find the solution to the case without the non-Newtonian term and then put the solution u_1 into the non-Newtonian term.

2 The Bending of Light by the Sun

The successful prediction of the bending of light by the Sun was the triumph that elevated Einstein's General Relativity to the point that it got the world's attention. In this problem you will calculate the bending of light by the Sun and see if you can get the correct answer by using Newtonian physics and the principle of equivalence - so that GR was not needed - and by dimensional analysis. The mass of the Sun is $M_\odot = 1.99 \times 10^{33}$ gm. The radius of the Sun is $R_\odot = 6.96 \times 10^{10}$ cm. You can use the impact approximation approach from a previous problem set.

2.1 Bending According to Equivalence Principle

Use the principle of equivalence - gravity = acceleration - and Newtonian physics to calculate the angle by which a light from a distant star (effectively infinity) nearly grazing the Sun's surface is bent by the time it passes us on Earth (also treat this as effectively infinity). If you need it in this approach, because of the spherical symmetry, angular momentum is conserved relative to the center of the Sun.

2.2 Dimensional Analysis

What two numbers do you have? Make two distances by calculating the black hole radius of the Sun and compute an angle by taking the ratio of two distances. How close is your answer to what you got in 3.1?

2.3 Use the Schwarzschild metric

Use the Schwarzschild metric, i.e for a spherically symmetric mass outside of the mass:

$$ds^2 = (1 - 2GM/rc^2)c^2dt^2 - (1 - 2GM/rc^2)^{-1}dr^2 + r^2d\Omega^2$$

and calculate the angle for the bending of starlight grazing the Sun. Use the impact parameter approximation to determine the geodesic deviation from a straight line.

3 The Gravitational Redshift

This problem will give you some familiarity with the gravitational redshift.

3.1 Formula

Give a formula for the fractional change in frequency (energy) of photons when its gravitational potential changes by an amount $\delta\phi$. (Hint: Use the equivalence principle to calculate the fractional difference in detected frequencies or energies. Or use the Schwarzschild metric.)

3.2 Gravitational Redshift in a Tower

Suppose you have a source of 14.4 keV photons emitted by an ^{57}Fe source on a tower of height 22.6 meters (74 ft). You detect the frequency of the emitted photons with a detector on the same level as the emitter and with another detector at the base of the tower. What frequency or energy difference do you observe.

Pound and Rebka (1960) performed this experiment and detected the gravitational shift using the Mössbauer effect What speed would the detector would give the same effect as the gravitational redshift? (Give the answer in m/s.)

3.3 Redshift from Sun

What is the fractional difference (redshift) for photons leaving the surface of the Sun?

3.4 Redshift from Center of the Earth

Suppose the Earth were transparent, so we could see the light emitted by the hot, glowing center of the Earth. What would be the redshift of this light received at the surface of the Earth? For this calculation, pretend that the Earth is a sphere of uniform density, and ignore the rotation of the Earth.

4 The Retardation of Light - Shapiro Time Delay

The curvature of spacetime causes a delay or retadardation in the arrivial time of signals when a massive body is placed near the line of sight. I.I. Shapiro realized this and an experimental test was done using a radio transponder landed on Mars as the orbit of Mars took Mars around behind the Sun.

Calculate the time delay caused for the light coming just past the limb (solar radius - calculate from impact parameter b and evaluate for the solar radius) relative to he Sun not being there. Use the Schwarzschild metric. You can approximate Mars and Earth as being very far from the Sun.

5 Black Hole Radius

Calculate the black hole radius of the Earth. ($G = 6.67 \times 10^{-8}$ cm^{2}gm^{-1}sec^{-2})

a) What is the radius for which the escape velocity would be the speed of light, if all the mass (5.98×10^{27} gm) associated directly with the Earth were concentrated in an infinitesimally small region (that is, smaller than the black hole radius)?

b) What is the radius for which the gravitational potential energy just matches the rest mass (mc^2) energy of a test particle?

c) What would be the speed at these radii of a test particle that came from rest at infinity? Calculate this using Newtonian physics. What would be the velocity in the case of a treatment of special relativity? Remember that $E = m(1-(v/c)^2)^{-1/2}$.

d) What does the Schwarzschild metric give for the blackhole radius